Title	On the Riesz Sums of Some Arithmetical Functions (P-Adic L-Functions and Algebraic Number Theory)
Author(s)	KANEMITSU, SHIGERU
Citation	数理解析研究所講究録 (1981), 411: 109-120
Issue Date	1981-01
URL	http://hdl.handle.net/2433/102416
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
411 巻 1981 年 109-120

On the Riesz sums of some arithmetical functions

S. Kanemitsu (Kyushu Univ.)

1. Introduction. The Riesz sums of arithmetical functions a_n referred to in the title, to be defined below, are frequently used mostly for finding asymptotic formulas for the summatory functions of a_n though they themselves are of interest. Let $\kappa \geqq 0$ be a real number and $\{\lambda_n\}_{n=1}^{\infty}$, $\{\ell_n\}_{n=1}^{\infty}$ be arbitrary sequences of real numbers strictly increasing to infinity such that $\lambda_1 \geqq 1$, $\ell_1 \geqq 0$. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence of complex numbers. Then we define

$$(1) \qquad \begin{cases} A_{\kappa}^{\kappa}(x) = \dfrac{1}{\Gamma(\kappa+1)} \displaystyle\sum_{\lambda_n \leqq x} a_n (x - \lambda_n)^{\kappa}, \\ A_{\ell}^{\kappa}(x) = \dfrac{1}{\Gamma(\kappa+1)} \displaystyle\sum_{\ell_n \leqq x} a_n (x - \ell_n)^{\kappa}, \end{cases}$$

and call $A_{\lambda}^{\kappa}(x)$ (resp. $A_{\ell}^{\kappa}(x)$) the Riesz sum of order κ, of the second (resp. first) kind, associated with the series $\sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ (resp. $\sum_{n=1}^{\infty} a_n e^{-\ell_n s}$). For the special choice of λ (resp. ℓ), i.e. for $\lambda_n = n$ or $N\mathfrak{a}$ (resp. $\ell_n = \log n$ or $\log N\mathfrak{a}$) we denote the corresponding $A_{\lambda}^{\kappa}(x)$ (resp. $A_{\ell}^{\kappa}(\log x)$) by $A_a^{\kappa}(x)$ (resp. $A_{\ell}^{\kappa}(x)$) and call it the arithmetic (resp. logarithmic) Riesz sum of order κ, associated with the series $\sum_{n=1}^{\infty} a_n n^{-s}$ (resp. $\sum_{\mathfrak{a}} a_{\mathfrak{a}} (N\mathfrak{a})^{-s}$.
For the general theory of typical means of Riesz we refer to Chandrasekharan & Minakshisundaram [3] and Hardy & Riesz [9], and for some basic results on Riesz sums, see Karamata [12].

Our previous intension was only to establish a general procedure by which one can obtain asymptotic formulas for $A_{\lambda}^{\kappa}(x)$ and $A_{\ell}^{\kappa}(x)$ for <u>any</u> order $\kappa > 0$. But we can also deduce from such asymptotic formulas of integral order a general Tauberian theorem which shows its effect when applied to finding asymptotic formulas for the sums of those a_n whose generating functions contain as a factor the reciprocals of some zeta-functions because in such cases we can appeal to the zero-free regions, if any, of the zeta-functions. A prototype of this kind of idea can be seen e. g. in a recent work of Ivić [10]. It should be added that our theorems are useful to obtain asymptotic formulas with error terms estimated uniformly with respect to some additional parameters other than x, e. g. with respect to $\Delta = N\mathfrak{f}\cdot|d|$ or to k, as will be illustrated in examples below.

Notations. ε denotes any positive number, η a suitable positive constant whose meaning is apparent from the context, $A, A_1, \ldots$ denote positive constants depending on $n = [K:Q]$, not necessarily the same at each occurence.

2. Asymptotic formulas for Riesz sums and Tauberian theorems.

Theorem 1. Let $\sigma_a > 0$ (which we may assume without loss of generality) be the abscissa of absolute convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n\lambda_n^{-s} = F(s)$, and for $b > \sigma_a$ let $B(b)$ denote $\sum_{n=1}^{\infty} |a_n|\lambda_n^{-b}$. Suppose that $F(s)$ can be continued analytically to a function meromorphic in some region R' extending vertically from top to bottom of the complex plane and bounded on the left by a piecewise smooth Jordan curve $\Gamma : \sigma = f(t)$, $0 < f(t) < b$, and that all the poles of $F(s)$ lying in R' are contained in a finite part of R' and are not on Γ. Take a subregion R bounded by three line segments $\overline{AB}$, $\overline{BC}$, $\overline{DA}$ and that part CD of Γ with $|t| \leqq T$ and T is so large that all the poles of $F(s)$ are contained in R. Suppose that $F(s)$ satisfies the following growth conditions: there is a constant $\mu < \kappa + 1$ such that

(i) $\quad F(s) = O(T^{\mu+\varepsilon}) \quad$ on $\overline{AD}$ and $\overline{BC}$;

(ii) $\quad F(s) = O(|t|^{\tau} V(t)) \quad$ on Γ if $|t| \geqq t_0$;

(iii) $\quad F(s) = O(W(f(t), t_0)) \quad$ on Γ if $|t| \leqq t_0$,

where V, W are positive, integrable and $V(y) = o(y^{\varepsilon})$ as $y \to \infty$, and $t_0 > 0$ is some constant. Then:

I. If $f(t)$ is given by ($\mathcal{L} = \log \Delta(|t| + 2)$)

$$\left.\begin{aligned} f(t) &= \beta - \psi(t) \geqq \eta > 0 \\ \psi(t) &= A\mathcal{L}^{-a'}(\log \mathcal{L})^{-b'} \end{aligned}\right\} \quad |t| \geqq t_0$$

with constants a', b', A, Δ, β such that $a' \geqq 0$, $b' \geqq 0$, $A > 0$, $\Delta \geqq 1$, we have, by taking $T = x^{\alpha}$ with a constant $\alpha > 0$, to be fixed when applied to specific problems, for any $\kappa > \tau$

$$A_{\lambda}^{\kappa}(x) = Q_{\kappa}(x) + R_{\lambda,I}^{\kappa}(x), \tag{2}$$

provided that $\log \Delta \ll (\log x)^{1/(1+a')-\eta}$ for some $\eta > 0$, where $Q_{\kappa}(x)$ is the sum of the residues of $\Gamma(s)F(s)x^{s+\kappa}/\Gamma(s+\kappa+1)$

in R,

$$(3) \qquad R^{\kappa}_{\lambda,\mathrm{I}}(x) = O(x^{\kappa+b}(x^{(\mu+\varepsilon-\kappa-1)\alpha} + x^{-\alpha\kappa}B(b)) + O(x^{\kappa+u}W(\beta,\Delta)) + O(x^{\kappa+\beta}\delta_A(x)),$$

$$(4) \qquad \delta_A(x) = \delta_{A,a',b'}(x) = \exp(-A(\log x)^{1/(1+a')}(\log\log x)^{-b'/(1+a')}),$$

and $u = \max_{|t| \leqq t_0} f(t)$.

II. If $f(t)$ is given by

$$f(t) = \beta \ (= \text{constant}),$$

then, with a constant $\alpha > 0$, we have

$$(5) \qquad A^{\kappa}_{\lambda}(x) = Q_{\kappa}(x) + R^{\kappa}_{\lambda,\mathrm{II}}(x),$$

where

$$(6) \qquad R^{\kappa}_{\lambda,\mathrm{II}}(x) = O(x^{b+\kappa}(x^{(\mu+\kappa-\varepsilon-1)\alpha} + x^{-\alpha\kappa}B(b)) + O(x^{\kappa+\beta}(1 + W(\beta))).$$

Corollary 1. Suppose that the conditions of Theorem 1 are satisfied and let q be the maximum of the real parts of poles of $F(s)$ in R, and r the maximal order of poles with real parts q, and finally define θ to be 1 or 0 according as $F(s)$ has a pole in R' or not, and θ' to be 1 or 0 according as $a_n \geqq 0$ for all $n \in \mathbf{N}$ or not. Then

$$(7) \qquad A^{0}_{\lambda}(x) = \theta Q_0(x) + \theta' O\Big(\sum_{x < \lambda_n \leqq x+\kappa y} |a_n|\Big) + \theta O(yx^{q-1}\log^{r-1}x) + O(y^{-\kappa}R^{\kappa}_{\lambda,i}(x)),$$

where $\kappa \in \mathbf{N}$ and $i = \mathrm{I}$ or II according to the choice of $f(t)$.

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied. Then:

III. If $f(t)$ is taken in the same way as in I, we have for any $\kappa > \tau$ (taking $T = e^{\alpha x}$)

$$(9) \qquad A_\ell^\kappa(x) = P_\kappa(x) + O(R_{\lambda,\mathrm{III}}^\kappa(x)),$$

where $P_\kappa(x)$ denotes the sum of the residues of $\frac{F(s)}{s^{\kappa+1}} e^{xs}$ in R' and

$$(10) \qquad R_{\lambda,\mathrm{III}}^\kappa(x) = O(e^{bx}(e^{(\mu+\varepsilon-\kappa-1)\alpha x} + e^{-\alpha\kappa x}B^*(b)) + O(e^{ux}W(\beta,\Delta)) + O(e^{\beta x}\delta_A(x)),$$

with

$$B^*(b) = \sum_{n=1}^{\infty} |a_n| \exp(-\ell_n b).$$

IV. If $f(t)$ is taken in the same way as in II, we have for any $\kappa > \tau$

$$(11) \qquad A_\ell^\kappa(x) = P_\kappa(x) + R_{\lambda,\mathrm{IV}}^\kappa(x),$$

where

$$(12) \qquad R_{\lambda,\mathrm{IV}}^\kappa(x) = O(e^{bx}(e^{(\mu+\varepsilon-\kappa-1)\alpha x} + e^{-\alpha\kappa x}B^*(b)) + O(e^{\beta x}(1 + W(\beta)).$$

Corollary 2. Under the assumptions of Theorem 1 and notations of Corollary 1 we have

$$(13) \qquad A_\ell^0(x) = P_0(x) + \theta' O(\sum_{x < \lambda_n \leqq x+\kappa y} |a_n|) + \theta O(ye^{qx}x^{r-1}) + O(y^{-\kappa}R_{\lambda,i}^\kappa(x)),$$

where $\kappa \in \mathbf{N}$ and $i = \mathrm{III}$ or IV according to the choice of $f(t)$.

Corollary 2'. If the asymptotic formula for $\sum_{\lambda_n \leqq x} |a_n|$ is of the following shape:

$$(14) \qquad \sum_{\lambda_n \leqq x} |a_n| = x^{q'} \log^{r'-1} x \,(C + o(1)),$$

then

$$(15) \qquad A_\ell^0(x) = \theta P_0(\log x) + \theta' O(\delta x^{q'} \log^{r'-1} x) + \theta O(\delta^{1/\kappa} x^q \log^{r-1} x)$$

$+ O(\delta^{-1}x^{-\alpha\kappa}(b - q')^{[r']+1} + x^{b+(\mu+\varepsilon-\kappa-1)\alpha}) + O(\delta^{-1}x^{u}W(\beta,\Delta)) + O(x^{\beta}\delta).$

For proofs of these results we use known information on the gamma-function, etc. contained in [27]. For Tauberian theorems, see, e. g. Postnikov [18].

3. Examples. We now state some examples.

Example 1 (For general reference regarding this examle, cf. [13], [16], [25]). Let K be an algebraic number field of degree n, fixed throughout, with discriminant d. Let $\mathcal{O} = \mathcal{O}_K$ be the ring of algebraic integers in K and $\mathfrak{f}$ be an arbitrary, fixed non-zero ideal of $\mathcal{O}$. Let $A_{\mathfrak{f}}$ be the group of all the fractional ideals with numerators and denominators relatively prime to $\mathfrak{f}$, and $H^*(\mathfrak{f})$ denote the ray class group of K, i.e. the quotient of $A_{\mathfrak{f}}$ by the group $S_{\mathfrak{f}}$ of principal ideals (α) with totally positive α such that $\alpha \equiv 1 \pmod{\mathfrak{f}}$. We define the Möbius function $\mu(\mathfrak{a})$ on ideals in the same manner as in the rational case and for $\mathcal{L} \in H^*(\mathfrak{f})$ we put

$$M(x,\mathcal{L}) = \sum_{N\mathfrak{a} \leqq x,\ \mathfrak{a}\in\mathcal{L}} \mu(\mathfrak{a}).$$

Then we have

Theorem 3 (A version of the Siegel-Walfisz prime ideal theorem). If $\Delta = N\mathfrak{f}\cdot|d| \ll \log^A x$, with an arbitrary constant A, however large it may be, we have

$$M(x,\mathcal{L}) = O_{n,A}(x\exp(-a\sqrt{\log x})), \tag{16}$$

with a constant $a = a(n,A) > 0$ depending at most on n and A, where the O-constant depends at most on n and A.

The proofs goes on the similar lines as those of Lemma 5 in [4] using Fogels' results on the zero-free region of $L(s,\chi)$ ([6], [7]). The details are omitted here.

Remark 1. The form of the reducing factor δ or of the zero-free region can be generalized using Karamata's slowly oscillating (or regularly varying) functions (cf. Seneta [21]).

Remark 2. Now that we have a uniform estimate for $M(x,\mathfrak{L})$, we can deduce asymptotic formulas by the "hyperbola method" (this naming is due to Ahern [1]) for those arithmetical functions defined on ideals which are given as the Dirichlet convolution of some familiar arithmetical functions whose asymptotic formulas are known. The details will appear elsewhere.

Remark 3. If we do not adhere to the uniformity of the estimate, we may appeal to the zero-free regions for the zeta-functions of algebraic number fields obtained by Mitsui [15] and Sokolovskiĭ [23] with the aid of which we can take $a' = 2/3$, $b' = 1/3$ in (4) and we shall have

$$M(x,\mathfrak{L}) = O(x \exp(-a(\log x)^{3/5}(\log\log x)^{-1/5}), \tag{16'}$$

$a = a(\Delta,n)$ being a constant depending on K. In fact, this results from the following remark: Take the class field corresponding to the ray class group $H^*(\mathfrak{f}_\chi)$, where $\mathfrak{f}_\chi | \mathfrak{f}$ is the conductor of χ. Then from the decomposition theorem in class field theory we have

$$\zeta_L(s) = \zeta_K(s) \prod_{\chi \neq \chi_0} L(s,\chi), \tag{17}$$

where on the RHS the L-function associated with χ appears. In (17), $\zeta_L(s)$ and $\zeta_K(s)$ have the zero-free regions described above, and $L(s,\chi)$ are regular there, so that all the L's on the RHS of (17) have zero-free regions of the same form. Now Corollay 2' completes the proof.

Finally note that appealing to the results of Goldstein [8], we could obtain an estimate for the error term uniformly w. r. t. n also.

Example 2. Let us consider two arithmetical functions appearing in [14], the first relating to $\frac{\mu(\mathfrak{a})}{N\mathfrak{a}}$ and the second, to Euler's function $\phi(n)$. For $\kappa \in \mathbf{N}$ set

$$\mathcal{M}_{\ell}^{\kappa}(x) = \frac{1}{\kappa!} \sum_{N\mathfrak{a} \leqq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}}\left(\log \frac{x}{N\mathfrak{a}}\right)^{\kappa} .$$

Since $F(s) = \zeta_K(s+1)^{-1}$, we have

$$P_K(\log x) = \sum_{n=1}^{\kappa} \frac{a_n}{(\kappa - n)!}(\log x)^{\kappa-n},$$

where a_n are determined by

$$\zeta_K(s)^{-1} = \sum_{n=1}^{\infty} a_n (s-1)^n.$$

Since $\zeta_K(s) = L(s,\chi_0)$ for $\mathfrak{f} = \mathcal{O}$, it follows from the analysis of L's and Corollary 2' that

$$\mathcal{M}_{\ell}^{\kappa}(x) = \sum_{n=1}^{\kappa} \frac{a_n}{(\kappa - n)!}(\log x)^{\kappa-n} + O(\delta_A(x)). \tag{18}$$

Note that (18) enables us to determine the power series coefficients of $\zeta(s)^{-1}$, hence those of $\zeta(s)$ using the generalized Euler constants γ_k defined by $(k \in \mathbf{N}\cup\{0\})$

$$\sum_{n \leqq x} \frac{\log^k n}{n} = \frac{1}{k+1} \log^{k+1} x + \gamma_k + O\left(\frac{\log^k x}{x}\right), \tag{19}$$

where

$$\gamma_k = k\int_1^{\infty} \frac{\psi(t)\log^{k-1} t}{t^2}\, dt \ (k \in \mathbf{N}),\ \gamma_0 = \gamma = \frac{1}{2} - \int_1^{\infty} \frac{\psi(t)}{t^2}\, dt, \tag{20}$$

the last being Euler's constant, thus giving another, more or less equivalent proof of Briggs & Chowla's result [2]. Actually, by (19) and the 2nd Möbius inversion formula we get a relation between $\mathcal{M}_{\ell}^{m+1}(x)$ and $\mathcal{M}_{\ell}^{n}(x)$ for $1 \leqq n \leqq m$, whence follows

$$a_{n+1} + \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \gamma_r a_{n-r} = 0,$$

or letting

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} A_n (s-1)^n,$$

we see that

(21) $$A_n = \frac{(-1)^n}{n!} \gamma_n .$$

Next, for $m \geqq 0$ and $\kappa > \frac{1}{2}$ let

$$\Phi_a^{\kappa}(x;m) = \frac{1}{\Gamma(\kappa+1)} \sum_{n \leqq x} \left(\frac{\phi(n)}{n}\right) n^m (x-n)^{\kappa}.$$

Then

(22) $$\Phi_a^{\kappa}(x;m) = \frac{\Gamma(m+1)}{\Gamma(\kappa+m+2)\zeta(2)} x^{m+\kappa+1} + \begin{cases} O(x^{m+\kappa}\delta_A(x)), \\ O(x^{m+\kappa-1/2+\varepsilon}), \end{cases}$$

the latter being valid on the assumption of the Riemann hypothesis (abbreviated: on the R. H.). Using this and writing $\Phi(x) = \Phi_a^0(x;1)$ $= \frac{3}{\pi^2} x^2 + E(x)$, we have by partial summation

$$\int_1^x E(t)\,dt = \begin{cases} O(x^2\delta) \\ O(x^{3/2+\varepsilon}) & \text{on the R. H.,} \end{cases}$$

which is, by a result of Segal [20], equivalent to

(23) $$\sum_{n \leqq x} E(n) = \frac{3x^2}{2\pi^2} + \begin{cases} O(x^2\delta) \\ O(x^{3/2+\varepsilon}) & \text{on the R. H.} \end{cases}$$

(23) gives 4.109 & 4.110 in [14]. Finally, write $\Phi_a^0(x;0) = \frac{6}{\pi^2} x$ $+ H(x)$. Then, using (22), we get

(24) $$H(x) = x^{-1}E(x) + O(\delta) = O((\log x)^{2/3}(\log\log x)^{1+\varepsilon})$$

by Saltykov's result [19]. Other results concerning ϕ will appear elsewhere.

Example 3 (Piltz's divisor problem). For the summatory function of $d_k(n)$ one can obtain an asymptotic formula of the same nature as that of Karacuba [11]. Note the slip on p. 481, ℓ.11.

References

[1] P. R. Ahern, The asymptotic distribution of prime ideals in ideal classes, Indag. Math. 26 (1964), 10-14.

[2] W. E. Briggs and S. Chowla, The power series coefficients of $\zeta(s)$, Amer. Math. Monthly, 62 (1955), 323-325.

[3] K. Chandrasekharan and S. Minakshisundaram, Typical means, Oxford Univ. Press, Oxford 1952.

[4] K. Chandrasekharan and Raghavan Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Ann. of Math. 76 (1962), 93-136.

[5] H. Davenport, On some infinite series involving arithmetical functions, Quart. J. Math. Oxford Ser. 8 (1937), 313-320 = The collected works of Harold Davenport, Vol IV, 1787-1794, Academic Press 1977.

[6] E. Fogels, On the zeros of Hecke's L-functions I, Acta Arith. 7 (1962), 87-106.

[7] ——"——, Über die Ausnahmenullstelle der Heckeschen L-Funktionen, Acta Arith. 8 (1963), 307-309.

[8] L. J. Goldstein, A generalization of the Siegel-Walfisz theorem, Trans. Amer. Math. Soc. 149 (1970), 417-429.

[9] G. H. Hardy and M. Riesz, The general theory of Dirichlet's series, Cambridge Math. Tracts No. 18, Cambridge Univ. Press, Cambridge 1915.

[10] A. Ivić, On the asymptotic formula for a generalization of von Mangoldt's function, Rend. Mat. (4) 10 (1977), 51-59.

[11] A. A. Karacuba, Uniform approximation of the remainder term in the Dirichlet divisor problem, Izv. Akad. Nauk SSSR, Ser.

Mat. 36 (1972), 457-483 = Math. USSR Izvestija, 6 (1972), 467-475.

[12] J. Karamata, Évaluation élémentaire des sommes typiques de Riesz de certaines fonctions arithmétiques, Acad. Serbe Sci. Publ. Inst. Math. 7 (1954), 1-40.

[13] E. Landau, Über Ideale und Primideale in idealklassen, Math. Z. 2 (1918), 52-154.

[14] D. S. Mitrinović and M. S. Popadić, Inequalities in number theory, Univ. of Niš, Belgrade 1978.

[15] T. Mitsui, On the prime ideal theorem, J. Math. Soc. Japan, 20 (1968), 233-247.

[16] Wł. Narkiewicz, Elementary and analytic theory of algebraic numbers, PWN, Warszawa 1974.

[17] K. Prachar, Primzahlverteilung, Springer Verl., Berlin-Heidelber-New York 1957.

[18] A. G. Postnikov, Tauberian theory and its applications, Trudy Mat. Inst. Steklov, No. 144, Nauka, Moskva 1979 = Proc. Steklov Inst. Math. 144 (1980).

[19] A. I. Saltykov, On Euler's function, Vestnik Moskov. Univ. Ser. I Mat. Meh. 1960, No. 6, 34-50 (Russian).

[20] S. L. Segal, A note on the average order of number-theoretic error terms, Duke Math. J. 32 (1965), 279-284.

[21] E. Seneta, Regularly varying functions, Lecture Notes in Math. No. 508, Springer Verl., Berlin-Heidelberg-New York 1976.

[22] R. A. Smith, The average order of a class of arithmetic functions over arithmetic progressions with applications

to quadratic forms, J. Reine Angew. Math. 317 (1980), 74-87.

[23] A. V. Sokolovskiĭ, A theorem on the zeros of the Dedekind zeta-functions and the distance between the "neighbouring" primes, Acta Arith. 8 (1968), 321-334 (Russian).

[24] G. Sri Rama Chandra Murty, Some contributions to the theory of r-free integers and associated sets of integers, Thesis, Andhra Univ. (India), 1979.

[25] Z. Suetuna, Kai-seki-teki-sei-sû-ron, Iwanami, Tokyo 1950.

[26] A. Walfisz, Weylsche Exponentialsummen in der neueren Zahlentheorie, VEB Deuscher Verlag der Wissenschaften, Berlin 1963.

[27] E. T. Whittaker and G. N. Watson, A course of modern analysis, 4th ed., Cambridge Univ. Press, Cambridge 1927.

Addenda. We add some more examples to which our theorems are applicable:

Example 4. The characteristic function of the set of M-void integers. Let M be a fixed subset of $\mathbf{N}$ with the least element ≥ 2. A positive integer n is called <u>M-void</u> (<u>M-leer</u>) if among the exponent of its canonical decomposition there do not appear any element of M. This includes the concepts of square-free, cube-free,..., k-free,... integers and others. Our theorems provide all the results contained on pp.66-106 in [24].

Example 5. Ivić's generalization of von Mangoldt's function.

Example 6. The characteristic function of the set of square-full integers, where a positive integer n is called <u>square-full</u> if $p^2|n$ for every prime factor p of n.

etc.